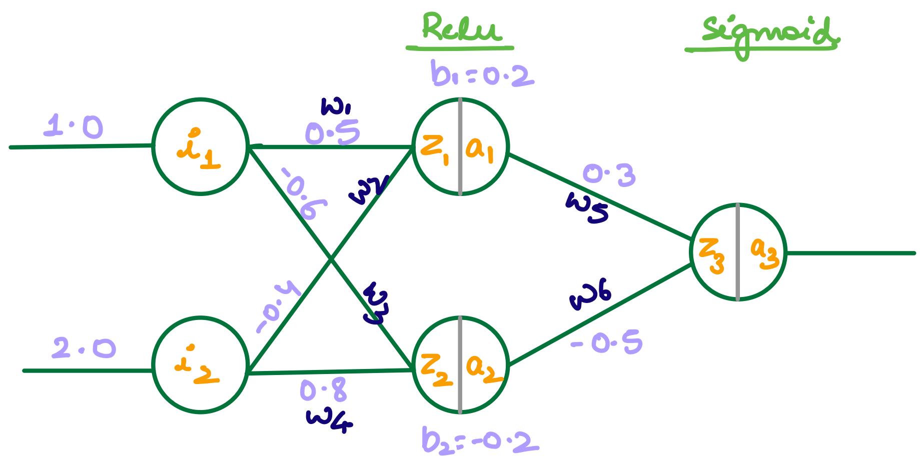
ML End Semester Nov 2023

4. Answer *any TWO* of the followings

[a] Consider a two-layer neural network used for binary classification. The network has an input layer with 2 neurons, a hidden layer with 2 neurons, and an output layer with 1 neuron. The activation function for the hidden layer is ReLU (Rectified Linear Unit), and for the output layer, it's a sigmoid function. The network is trained using the binary cross-entropy loss function and stochastic gradient descent (SGD) with a learning rate of 0.01. The initial weights and biases are as follows: Weights from input to hidden layer: $W_1 = [[0.5, -0.6], [-0.4, 0.8]]$, Bias for hidden layer: $b_1 = [0.2, -0.2]$, Weights from hidden to output layer: $W_2 = [0.3, -0.5]$, Bias for output layer: $b_2 = 0$. Consider the network is trained with a single training sample ($X = [1.0, 2.0]$, $Y = 0$). Perform the forward pass to calculate activations at hidden layer and output layer, and then compute the loss. [4] [CO2]

[b] Consider the neural network in 4[a] again and perform the backpropagation to update the weights and biases. Calculate the updated weights W_1, W_2, and biases b_1, b_2 after one iteration. Show your calculations for the forward pass, loss calculation, and backpropagation steps. [4] [CO2]



$z_1 = i_1 w_1 + i_2 w_2 + b_1 \quad = 1 * 0.5 + 2 * (-0.4) + 0.2 \quad = -0.1$

$a_1 = \text{Relu}(z_1) = 0$

$$\text{Relu}(z_1) = \begin{cases} 0 & \text{if } z_1 \le 0 \\ z_1 & \text{if } z_1 > 0 \end{cases}$$

$z_2 = i_1 w_3 + i_2 w_4 + b_2$

$\quad = 1 * (-0.6) + 2 * (0.8) - 0.2 \quad = 0.8$

$a_2 = \text{Relu}(z_2) = 0.8$

$z_3 = a_1 w_5 + a_2 w_6 \quad = 0 * 0.3 + 0.8 * (-0.5)$

$\quad = -0.4$

$$\text{Sigmoid}(x) = \frac{1}{1+e^{-x}}$$

$a_3 = \text{Sigmoid}(z_3) = \text{Sigmoid}(-0.4) = \frac{1}{1+e^{0.4}} = 0.401$

loss fx^n being used here is Binary Cross Entropy.

$$\boxed{BCE = -\sum_{i=1}^{m} \left(y^{(i)} \log \hat{y}^{(i)} + (1-y^{(i)}) \log (1-\hat{y}^{(i)}) \right)}$$

Since there is only one example in question, so $m=1$

we use 'J' to represent loss.

$$J = -\left(y \log \hat{y} + (1-y) \log (1-\hat{y}) \right)$$
$$= -\left(0 \cdot \log 0.401 + (1-0) \log (1-0.401) \right)$$
$$= -\log 0.599 = 0.2225$$

$\hat{y}$ is actually the output from the output layer i.e. a_3

$\hat{y} = a_3$

Backpropagation

$$* \quad \frac{\partial J}{\partial w_5} = \frac{\partial J}{\partial a_3} \cdot \frac{\partial a_3}{\partial z_3} \cdot \frac{\partial z_3}{\partial w_5}$$

(1) (2) (3)

(1) $\frac{\partial J}{\partial a_3}$?

$$J = -\left(y \log \hat{y} + (1-y) \log (1-\hat{y}) \right)$$

a_3 is actually $\hat{y}$

$$\text{so, } J = -\left(y \log a_3 + (1-y) \log (1-a_3) \right)$$

$$\frac{\partial J}{\partial a_3} = -\left(\frac{y}{a_3} - \frac{(1-y)}{(1-a_3)} \right) = -\left[\frac{y(1-a_3) - a_3(1-y)}{a_3(1-a_3)} \right]$$

$$= -\left[\frac{y - \cancel{ya_3} - a_3 + \cancel{ya_3}}{a_3(1-a_3)} \right]$$

$$\frac{\partial J}{\partial a_3} = \frac{a_3 - y}{a_3(1-a_3)}$$

(2) $\frac{\partial a_3}{\partial z_3}$?

$$a_3 = \frac{1}{1+e^{-z_3}}$$

$$\frac{\partial a_3}{\partial z_3} = (a_3)(1-a_3)$$

$$y = \frac{1}{1+e^{-x}} = (1+e^{-x})^{-1}$$

$$\frac{\partial y}{\partial x} = \frac{-1}{(1+e^{-x})^2}(-e^{-x})$$

$$\frac{\partial y}{\partial x} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$\frac{\partial y}{\partial x} = \frac{1}{(1+e^{-x})} \cdot \frac{e^{-x}}{(1+e^{-x})}$$

$$\frac{\partial y}{\partial x} = \left(\frac{1}{1+e^{-x}}\right)\left(1 - \frac{1}{1+e^{-x}}\right)$$

$$\frac{\partial y}{\partial x} = y(1-y)$$

Remember this

(3) $\frac{\partial z_3}{\partial w_5}$

$$z_3 = a_1 w_5 + a_2 w_6$$

$$\frac{\partial z_3}{\partial w_5} = a_1$$

$$\frac{\partial J}{\partial w_5} = (1)\,(2)\,(3)$$

$$= \underbrace{\frac{a_3 - y}{a_3(1-a_3)}}\ \underbrace{(a_3)(1-a_3)}\ \underbrace{a_1} \qquad = (a_3 - y)a_1$$

$$w_5 = w_5 - \eta \frac{\partial J}{\partial w_5}$$

$$= w_5 - \eta (a_3 - y) a_1$$

$$= 0.3 - (0.01)(0.401 - 0)\, 0 \qquad = 0.3\ \checkmark$$

$$* \quad \frac{\partial J}{\partial w_6} = \frac{\partial J}{\partial a_3} \cdot \frac{\partial a_3}{\partial z_3} \cdot \frac{\partial z_3}{\partial w_6}$$

(1) (2) (4)

(1) and (2) will remain same, only (4) will change

④ $\dfrac{\partial z_3}{\partial w_6}$?

$z_3 = a_1 w_5 + a_2 w_6$

$\dfrac{\partial z_3}{\partial w_6} = a_2$

$$\frac{\partial J}{\partial w_6} = ① \; ② \; ④$$

$$= \underbrace{\frac{a_3 - y}{a_3(1-a_3)}} \; \underbrace{(a_3)(1-a_3)} \; \underbrace{a_2} \qquad = (a_3 - y) a_2$$

$$w_6 = w_6 - \eta \frac{\partial J}{\partial w_6}$$

$$= w_6 - \eta (a_3 - y) a_2$$

$$= -0.5 - (0.01)(0.401 - 0)(0.8) = -0.5032 \checkmark$$

* $$\frac{\partial J}{\partial w_1} = \frac{\partial J}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w_1}$$

$$= \frac{\partial J}{\partial z_3} \cdot \frac{\partial z_3}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w_1}$$

$$= \frac{\partial J}{\partial a_3} \cdot \frac{\partial a_3}{\partial z_3} \cdot \frac{\partial z_3}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w_1}$$

① ② ⑤ ⑥ ⑦

already calculated

⑤ $\dfrac{\partial z_3}{\partial a_1}$?

$z_3 = a_1 w_5 + a_2 w_6$

$$\frac{\partial z_3}{\partial a_1} = w_5$$

⑥ $\frac{\partial a_1}{\partial z_1}$?

$$a_1 = \mathrm{Relu}(z_1) = \begin{cases} 0 & \text{if } z_1 \le 0 \\ z_1 & \text{if } z_1 > 0 \end{cases}$$

$$\frac{\partial a_1}{\partial z_1} = \begin{cases} 0 & \text{if } z_1 \le 0 \\ 1 & \text{if } z_1 > 0 \end{cases}$$

⑦ $\frac{\partial z_1}{\partial w_1}$?

$$z_1 = i_1 w_1 + i_2 w_2 + b_1$$

$$\frac{\partial z_1}{\partial w_1} = i_1$$

$$\frac{\partial J}{\partial w_1} = ① \; ② \; ⑤ \; ⑥ \; ⑦$$

$$= \underbrace{\frac{a_3 - y}{a_3(1-a_3)}} \; \underbrace{(a_3)(1-a_3)} \; \underbrace{w_5} \; \underbrace{\begin{matrix} 0 & \text{if } z_1 \le 0 \\ 1 & \text{if } z_1 > 0 \end{matrix}} \; \underbrace{i_1}$$

$$= (a_3 - y)(w_5)\begin{pmatrix} 0 & \text{if } z_1 \le 0 \\ 1 & \text{if } z_1 > 0 \end{pmatrix}(i_1)$$

$$= (0.401 - 0)(0.3)(0)(1.0) \quad = 0$$

$$w_1 = w_1 - \eta \frac{\partial J}{\partial w_1}$$

$$= 0.5 - (0.01)(0) = 0.5 \checkmark$$

$$* \quad \frac{\partial J}{\partial w_2} = \frac{\partial J}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w_2}$$

$$= \frac{\partial J}{\partial z_3} \cdot \frac{\partial z_3}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w_2}$$

$$= \underbrace{\frac{\partial J}{\partial a_3} \cdot \frac{\partial a_3}{\partial z_3} \cdot \frac{\partial z_3}{\partial a_1} \cdot \frac{\partial a_1}{\partial z_1}}_{\text{same as previous}} \cdot \frac{\partial z_1}{\partial w_2}$$

1 2 5 6 8

same as previous

8 $\quad \frac{\partial z_1}{\partial w_2}$?

$$z_1 = i_1 w_1 + i_2 w_2 + b_1$$

$$\frac{\partial z_1}{\partial w_2} = i_2$$

$$\frac{\partial J}{\partial w_2} = (1)(2)(5)(6)(8)$$

$$= \underbrace{\frac{a_3 - y}{a_3(1-a_3)}} \; \underbrace{(a_3)(1-a_3)} \; \underbrace{w_5} \; \underbrace{\begin{matrix} 0 & \text{if } z_1 \le 0 \\ 1 & \text{if } z_1 > 0 \end{matrix}} \; \underbrace{i_2}$$

$$= (a_3 - y)(w_5)\begin{pmatrix} 0 & \text{if } z_1 \le 0 \\ 1 & \text{if } z_1 > 0 \end{pmatrix}(i_2)$$

$$= (0.401 - 0)(0.3)(0)(2.0) = 0$$

$$w_2 = w_2 - \eta \frac{\partial J}{\partial w_2}$$

$$= -0.4 - (0.01)(0) = -0.4 \quad ✓$$

$$* \quad \frac{\partial J}{\partial w_3} = \frac{\partial J}{\partial a_2} \cdot \frac{\partial a_2}{\partial z_2} \cdot \frac{\partial z_2}{\partial w_3}$$

$$= \frac{\partial J}{\partial z_3} \cdot \frac{\partial z_3}{\partial a_2} \cdot \frac{\partial a_2}{\partial z_2} \cdot \frac{\partial z_2}{\partial w_3}$$

$$= \underbrace{\frac{\partial J}{\partial a_3}}_{1} \cdot \underbrace{\frac{\partial a_3}{\partial z_3}}_{2} \cdot \underbrace{\frac{\partial z_3}{\partial a_2}}_{9} \cdot \underbrace{\frac{\partial a_2}{\partial z_2}}_{10} \cdot \underbrace{\frac{\partial z_2}{\partial w_3}}_{11}$$

already calculated

9 $\frac{\partial z_3}{\partial a_2}$?

$$z_3 = a_1 w_5 + a_2 w_6$$

$$\frac{\partial z_3}{\partial a_2} = w_6$$

10 $\frac{\partial a_2}{\partial z_2}$?

$$a_2 = \text{Relu}(z_2) = \begin{cases} 0 & \text{if } z_2 \le 0 \\ z_2 & \text{if } z_2 > 0 \end{cases}$$

$$\frac{\partial a_2}{\partial z_2} = \begin{cases} 0 & \text{if } z_2 \le 0 \\ 1 & \text{if } z_2 > 0 \end{cases}$$

11 $\frac{\partial z_2}{\partial w_3}$?

$$z_2 = i_1 w_3 + i_2 w_4 + b_2$$

$$\frac{\partial z_2}{\partial w_3} = i_1$$

$$\frac{\partial J}{\partial \omega_3} = 1 \quad 2 \quad 9 \quad 10 \quad 11$$

$$= \underbrace{\frac{a_3 - y}{a_3(1-a_3)}} \; \underbrace{(a_3)(1-a_3)} \; \underbrace{\omega_6} \; \underbrace{\begin{matrix} 0 & \text{if } z_2 \le 0 \\ 1 & \text{if } z_2 > 0 \end{matrix}} \; \underbrace{i_1}$$

$$= (a_3 - y)\,(\omega_6) \begin{pmatrix} 0 & \text{if } z_2 \le 0 \\ 1 & \text{if } z_2 > 0 \end{pmatrix} (i_1)$$

$$= (0.401 - 0)\,(-0.5)\,(1)\,(1) = -0.2005$$

$$\omega_3 = \omega_3 - \eta \frac{\partial J}{\partial \omega_3}$$

$$= -0.6 - (0.01)(-0.2005) = -0.6 + 0.002005 = -0.597$$

✓

* $$\frac{\partial J}{\partial \omega_4} = \frac{\partial J}{\partial a_2} \cdot \frac{\partial a_2}{\partial z_2} \cdot \frac{\partial z_2}{\partial \omega_4}$$

$$= \frac{\partial J}{\partial z_3} \cdot \frac{\partial z_3}{\partial a_2} \cdot \frac{\partial a_2}{\partial z_2} \cdot \frac{\partial z_2}{\partial \omega_4}$$

$$= \underbrace{\underset{1}{\frac{\partial J}{\partial a_3}} \cdot \underset{2}{\frac{\partial a_3}{\partial z_3}} \cdot \underset{9}{\frac{\partial z_3}{\partial a_2}} \cdot \underset{10}{\frac{\partial a_2}{\partial z_2}}}_{\text{same as earlier}} \cdot \underset{12}{\frac{\partial z_2}{\partial \omega_4}}$$

12 $\frac{\partial z_2}{\partial \omega_4}$?

$$z_2 = i_1 w_3 + i_2 w_4 + b_2$$

$$\frac{\partial z_2}{\partial \omega_4} = i_2$$

$$\frac{\partial J}{\partial w_4} = ① \; ② \; ⑨ \; ⑩ \; ⑫$$

$$= \underbrace{\frac{a_3 - y}{a_3(1-a_3)}} \; \underbrace{(a_3)(1-a_3)} \; \underbrace{w_6} \; \underbrace{\begin{matrix} 0 & \text{if } z_2 \le 0 \\ 1 & \text{if } z_2 > 0 \end{matrix}} \; \underbrace{i_2}$$

$$= (a_3 - y)\,(w_6) \begin{pmatrix} 0 & \text{if } z_2 \le 0 \\ 1 & \text{if } z_2 > 0 \end{pmatrix} (i_2)$$

$$= (0.401 - 0)\,(-0.5)\,(1)\,(2) = -0.401$$

$$w_4 = w_4 - \eta \frac{\partial J}{\partial w_4}$$

$$= 0.8 - (0.01)(-0.401) = 0.8 + 0.00401 = 0.80401$$

✓

Reference Link : https://mattmazur.com/2015/03/17/a-step-by-step-backpropagation-example/